Feb 4

$\star$ $(\ln x)' = \frac{1}{x}$ $\quad x \neq 0$

proof

$$\frac{d \ln x}{dx} = \lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{\ln\left(1 + \frac{\Delta x}{x}\right)}{\Delta x / x} \cdot \frac{1}{x} \right]$$

$$= \frac{1}{x} \lim_{u \to 0} \left[\frac{\ln(1+u)}{u} \right]$$

Let $u = \frac{\Delta x}{x}$

when $\Delta x \to 0$

then $u \to 0^{\pm}$

depending on whether x is positive or negative

Example 4.1.6. Discuss the differentiability of $f(x) = |x|$. ← this is continuous function

$= \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$

(Rmk: A differentiable function is continuous but not vice versa)

Solution. For $x_0 > 0$,

$$\frac{dx}{dx} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x) - x_0}{\Delta x} = 1.$$

For $x_0 < 0$,

$$\frac{d(-x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} == \lim_{\Delta x \to 0} \frac{-(x_0 + \Delta x) - (-x_0)}{\Delta x} = -1.$$

For $x_0 = 0$.

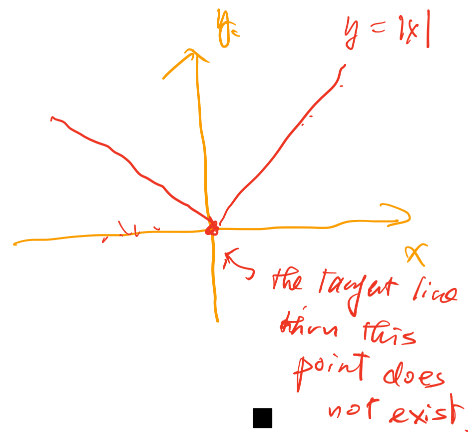
$$\lim_{\Delta x \to 0+} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0+} \frac{\Delta x}{\Delta x} = 1.$$

$$\lim_{\Delta x \to 0-} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0-} \frac{-\Delta x}{\Delta x} = -1.$$

$1 \neq -1$, so f is not differentiable at $x = 0$. So,

$\lim_{\Delta x \to 0}$ of $\left|\frac{\Delta f}{\Delta x}\right|_{x=0}$ doesn't exist

$$(|x|)' = \begin{cases} 1 & \text{if } x > 0, \\ \text{undefined} & \text{if } x = 0. \\ -1 & \text{if } x < 0, \end{cases}$$



■

4.2 Properties of derivatives

4.2.1 Differentiation and Continuity

Proposition 1. $f(x)$ is differentiable at $x = x_0 \implies f(x)$ is continuous at $x = x_0$.

Proof. Suppose $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists, then

$$\begin{aligned} \lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \right) \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) \\ &= f'(x_0) \cdot 0 = 0. \end{aligned}$$

0 if f is continuous at x_0

So, $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (f(x) - f(x_0)) + \lim_{x \to x_0} f(x_0) = 0 + f(x_0) = f(x_0)$, that is, $f(x)$ is continuous at x_0. □

The converse is not true. For example, let $f(x) = |x|$. It is not differentiable at $x = 0$ but is continuous at $x = 0$.

Exercise 4.2.1. Let

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x \geq 1 \\ 1 - x, & \text{if } x < 1 \end{cases}$$

(a) Show that $f(x)$ is continuous at $x = 1$.

(b) Show that $f(x)$ is differentiable everywhere except $x = 1$, and

$$f'(x) = \begin{cases} 2x, & \text{if } x > 1 \\ \text{undefined}, & \text{if } x = 1 \\ -1, & \text{if } x < 1 \end{cases}$$

4.2.2 Derivative and Arithmetic Operation

Theorem 2. *If $f(x)$ and $g(x)$ are differentiable function, then*

(1) *Sum rule:* $(f+g)'(x) = f'(x) + g'(x)$.

(2) *Difference rule:* $(f-g)'(x) = f'(x) - g'(x)$.

$\times\ f' \cdot g'$

(Leibniz rule) (3) *Product rule:* $\boxed{(fg)'(x) = f'(x)g(x) + f(x)g'(x).}$

(4) *Quotient rule:* $\boxed{\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.}$

Derive via the Leibniz rule + chain rule (§ 4.3)

Proof. (1)

$$\begin{aligned}
(f+g)'(x) &= \lim_{\Delta x \to 0} \frac{(f+g)(x+\Delta x) - (f+g)(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) + g(x+\Delta x) - (f(x) + g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x).
\end{aligned}$$

$f(x+\Delta x)\dfrac{g(x+\Delta x)-g(x)}{\Delta x}$

$+\; g(x)\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

(3)

$$
\begin{aligned}
(fg)'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x+\Delta x)g(x)+f(x+\Delta x)g(x)-f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x+\Delta x)g(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x)-f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left(f(x+\Delta x)\cdot\frac{g(x+\Delta x)-g(x)}{\Delta x}\right)+\lim_{\Delta x\to 0}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\cdot g(x)\right)\\
&= \lim_{\Delta x\to 0}f(x+\Delta x)\cdot\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\cdot\lim_{\Delta x\to 0}g(x)\\
&= f(x)g'(x)+f'(x)g(x).
\end{aligned}
$$

sum rule for limits

product rule for limits

Remark. Here we used:

$$g(x) \text{ is differentiable at } x \quad\Rightarrow\quad g(x) \text{ is continuous at } x$$

so, $\lim_{\Delta x\to 0} f(x+\Delta x)=f(x)$.

□

Exercise 4.2.2. Prove other rules using the first principle.

4.2.3 Derivative of Elementary Functions

Theorem 3 (Constant function).

$$\boxed{f(x)=k \quad\Rightarrow\quad f'(x)=0}$$

Proof.

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{k-k}{\Delta x}=0.$$

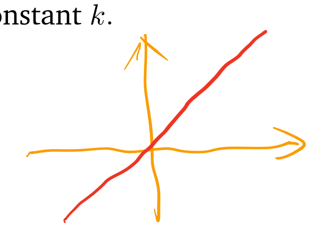
□

As a consequence, we have

$$\boxed{(kf(x))'=(k)'f(x)+kf'(x)=kf'(x),}\quad \text{for any constant } k.$$

E.g.) $f(x)=x$ $\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1$

$x'=1$



$f(x) = x^2$

$f' = (x \cdot x)'$ Leibniz $= x'x + xx'$

$= 2x$

Remark. It can also be proved by the first principle.

$(x^3)' = (x \cdot x^2)'$

$= 3x^2$

Theorem 4 (The Power Rule).

$$\boxed{(x^a)' = ax^{a-1},} \quad \textit{whenever it is well-defined, } a \in \mathbb{R}.$$

Proof. We will only prove the special case when n is an integer.
Recall

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}).$$

So

$$(x + \Delta x)^n - x^n = (x + \Delta x - x)((x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + \cdots + (x+\Delta x)x^{n-2} + x^{n-1}).$$

We have

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} &= \lim_{\Delta x \to 0} ((x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + \cdots + (x+\Delta x)x^{n-2} + x^{n-1}) \\
&= x^{n-1} + x^{n-2}x + \cdots xx^{n-2} + x^{n-1} = nx^{n-1}.
\end{aligned}$$

(Alternatively, use $x = 1$ & Leibniz rule

□

Example 4.2.1.

$$\begin{aligned}
(x^3)' &= 3x^2, & x \in \mathbb{R} \\
(\sqrt{x})' &= \frac{1}{2}x^{-\frac{1}{2}}, & x > 0. \quad & \textcolor{red}{\text{Caution: } x \text{ can not be } 0.} \\
(\sqrt[3]{x})' &= \frac{1}{3}x^{-\frac{2}{3}}, & x \neq 0. \quad & \textcolor{red}{\text{Caution: } x \text{ can be negative.}} \\
(x^{\frac{3}{2}})' &= \frac{3}{2}x^{\frac{1}{2}}, & x > 0.
\end{aligned}$$

$(\sqrt[3]{x})' = (x^{\frac{1}{3}})' = \frac{1}{3}x^{-\frac{2}{3}}$

when $x \neq 0$

Theorem 5 (Exponential function and Logarithmic function).

chain rule or power rule

motivation for the definition of e

$$\boxed{(e^x)' = e^x; \quad (a^x)' = a^x \ln a,} \qquad a > 0, a \neq 1, \textcolor{red}{x \in \mathbb{R}}.$$

$$\boxed{(\ln x)' = \frac{1}{x}; \quad (\log_a x)' = \frac{1}{x \ln a},} \qquad a > 0, a \neq 1, \textcolor{red}{x > 0}.$$

Proof. (Optional!)

$\ln\left(\frac{x+\Delta x}{x}\right)$

$\ln(ab) = \ln a + \ln b$

$-\ln b = \ln(b^{-1})$

$$(\ln x)' = \frac{1}{x} \iff \lim_{\Delta x\to 0}\frac{\ln(x+\Delta x)-\ln x}{\Delta x} = \frac{1}{x}$$

$$\iff \lim_{\Delta x\to 0}\frac{\ln(1+\frac{\Delta x}{x})}{\frac{\Delta x}{x}} = 1$$

$$\iff \boxed{\lim_{y\to 0}\ln(1+y)^{\frac{1}{y}} = 1,} \quad (\text{ let } y = \tfrac{\Delta x}{x})$$

$$\iff \lim_{y\to 0}(1+y)^{\frac{1}{y}} = e$$

← alternate definition of e

$y \to 0^+$ $\qquad$ $y \to 0^-$

$$\iff \lim_{z\to+\infty}\left(1+\frac{1}{z}\right)^z = e, \quad \lim_{z\to-\infty}\left(1+\frac{1}{z}\right)^z = e, \quad (\text{ let } z = \frac{1}{y}, \text{ definition of } e)$$

may be derived from each other using the fact that $\ln x$ and e^x are inverse functions of each other.

$$(e^x)' = \cancel{x}\ e^x \iff \lim_{\Delta x\to 0}\frac{e^{x+\Delta x}-e^x}{\Delta x} = e^x$$

$$\iff \lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x} = 1$$

$$\iff \lim_{y\to 0}\frac{y}{\ln(1+y)} = 1, \quad (\text{ let } y = e^{\Delta x}-1)$$

$$\iff \underbrace{\lim_{y\to 0}\frac{\ln(1+y)}{y}}_{} = 1$$

$$(\ln x)'\Big|_{x=1} = \frac{1}{x}\Big|_{x=1} = 1$$

□

$a^b = e^{b(\ln a)}$

Example 4.2.2.

1. $(\sqrt{x}+2^x-3\log_2 x)' = (\sqrt{x})'+(2^x)'-3(\log_2 x)' = \frac{1}{2}x^{-\frac{1}{2}}+2^x\ln 2-\frac{3}{x\ln 2}$

2. $\frac{d}{dx}(x^2e^x) = \frac{d}{dx}(x^2)\cdot e^x + x^2\cdot\frac{d}{dx}(e^x) = (2x+x^2)e^x$

3. $\left(\frac{\sqrt{x}}{3^x}\right)' = \left(x^{\frac{1}{2}}\cdot 3^{-x}\right)' =$ Leibniz rule $(x^{\frac{1}{2}})'\cdot 3^{-x} + x^{\frac{1}{2}}\cdot(3^{-x})'$

↓ power rule

$x > 0$

$$= \underbrace{\frac{1}{2}\cdot x^{-\frac{1}{2}}\cdot 3^{-x} + x^{\frac{1}{2}}(-\ln 3)\cdot 3^{-x}}_{}$$

□

$$\text{Quotient rule:} \quad \frac{(\sqrt{x})'3^x - \sqrt{x}(3^x)'}{(3^x)^2} = \frac{\frac{1}{2}x^{-\frac{1}{2}} \cdot 3^x - x^{\frac{1}{2}} \cdot 3^x \ln 3}{(3^x)^2} = \frac{\frac{1}{2}x^{-\frac{1}{2}} - x^{\frac{1}{2}} \ln 3}{3^x}$$

$$\text{Product rule:} \quad \left(\sqrt{x} \cdot \left(\frac{1}{3}\right)^x\right)' = \frac{1}{2}x^{-\frac{1}{2}}\left(\frac{1}{3}\right)^x + x^{\frac{1}{2}}\left(\frac{1}{3}\right)^x \ln\frac{1}{3} = \frac{\frac{1}{2}x^{-\frac{1}{2}} - x^{\frac{1}{2}} \ln 3}{3^x}$$

Exercise 4.2.3. Use two different methods to compute $\left(\frac{1-x^2}{\sqrt{x}}\right)'$.

Example 4.2.3. Suppose $f(x)$ and $g(x)$ are differentiable. Given $f(1) = 1$, $f'(1) = 2$, $g(1) = 3$, $g'(1) = 4$. Find the value of

$$\frac{d}{dx}\left(f(x)g(x)\right)$$

at $x = 1$.

Solution. By the product rule

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x).$$

At $x = 1$, the above is

$$f'(1)g(1) + f(1)g'(1) = 2 \times 3 + 1 \times 4 = 10.$$

■

Example 4.2.4. Suppose $f(x)$, $g(x)$, $h(x)$ are differentiable. Compute

$$\frac{d}{dx}\left(f(x)g(x)h(x)\right).$$

Solution.

$$\begin{aligned}\frac{d}{dx}\left(f(x)g(x)h(x)\right) &= \left(f(x)g(x)\right)\frac{d}{dx}h(x) + h(x)\frac{d}{dx}\left(f(x)g(x)\right)\\ &= f(x)g(x)h'(x) + h(x)(f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x))\\ &= f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x).\end{aligned}$$

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